CHAPTER 14

# *Kant and the Neo-Kantians on mathematics*

*Luca Oliva*

## Introduction

An important feature of late nineteenth- and earlytwentieth-century philosophy has been the emergence of mathematical questions. This trend can be easily traced back to Kant's theoretical philosophy, especially his account of mathematical knowledge based on intuitions. A legacy of Neo-Kantianism consists in offering variations on the Kantian model of intuitions, which appears consistent with current analytic readings of Kant (i.e., Analytic Kantianism). Roughly put, this reading introduces logical elements into mathematical reasoning in order, for instance, to make sense of numerical compositions. Neo-Kantians derived such elements from developing the Kantian notion of intuition, which has recently been seen in a similar way, namely, as carrying a logical connotation.

In the First Critique, Kant argues that the mathematical construction of concepts pertains to the synthesis of non-empirical intuitions (A 716/B 744). This synthesis is supposed to offer evidence that a priori judgments take the place of experience; and such evidence is supposed to rely on the exhibition of concepts *in concreto* (A 711/B 739). I believe that these suppositions are ultimately justified by the nature of intuitions, which I see as consistent with the logical, mathematical characterizations set forth by Hintikka and Parsons. Building on these references, my paper aims to show that Kant's intuitions behave like free variables which allow for abstract constructions such as numbers and basic operations on numbers, like addition.

I'll support my thesis by examining the parts of the *Transcendental Doctrine of Method* (A 709–38/B 738–766) which explicitly address the representation of the universal in the individuality of intuitions; and the parts of the *Transcendental Aesthetic* (A 22–36/B 37–53) along with those of the *Lectures on Metaphysics* (28:506–561) which characterize these intuitions as pointing at part-whole relation and magnitude. I'll then derive a consistent account of numbers which will show how intuitions evolve into homogeneous quanta mereologically combined in proportions. Such an evolution remains unconceivable unless we acknowledge the logical import that intuitions bear from the start.

Further evidence for this claim can be found in the Neo-Kantian account of numbers, which has thoroughly assumed the logical character of Kant's intuitions. On the one hand the Neo-Kantians do not recognize intuitions as a faculty independent of the understanding, on the other they introduce all the properties of spatiotemporal intuitions into their mathematical constructions. I'll prove this by carrying out a detailed examination, previously unattempted, of Rickert's account of logical and mathematical objects. Although Cassirer's and Natorp's logicism was ultimately rejected by Rickert, Neo-Kantians share Dedekind's constructionism, in which the theory of numbers replaces temporal intuitions with relational properties. This variation also supports my thesis.[1]

## Mathematical propositions

In his *Transcendental Doctrine of Method*, Kant characterizes mathematics as construction in intuition. He argues that mathematical concepts can be exhibited a priori by means of non-empirical intuitions (A 713/B 741). The significance of this part can hardly be underestimated, for Kant is giving a direct answer to the main question of his Critique, namely "How are a priori synthetic judgments possible?" (B 19).

The argument derives from the analysis of geometrical demonstrations. Kant notices that the truth-value of Euclid's propositions runs from one claim to another "through a chain of inferences guided throughout by intuition" (A 716/B 744). Any passage is both synthetic and evident. What really strikes Kant is that the evidence of those inferences does not come from experience, as is supposed to be in the case of synthetic claims. Even more surprising is that the synthesis carries on strict and not merely comparative universality. It hardly bears repeating that for Kant a strictly universal proposition states that *Ps* "cannot be otherwise" (A 1/B 3), namely, it is true in all possible worlds. A comparatively universal proposition, by contrast, states that it has never been disproved before, as is the case, for instance, with a valid generalization or with Hume's custom

[1] I would like to thank James Garson, Robert Hanna, and Paolo Parrini for all the corrections and critical comments. Many suggestions have come from Michael Friedman, Charles Parsons, Lisa Shabel, Daniel Sutherland, and Paul Teller. For the language I am grateful to Thomas Behr and Sharon Joyce.

induced inference. Thus instances are used to provide synthetic claims with *empirical evidence.* In this sense, Euclid's demonstrations genuinely challenge the classic distinction of analytic and synthetic that Kant thoroughly takes for granted.

In all judgments in which the relation of a subject to the predicate is thought ... this relation is possible in two different ways. Either the predicate to the subject A, as something which is (covertly) contained in this concept A; or outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgment analytic, in the other synthetic. (A 6–7/B 10)

Using current terminology, all judgments are reducible to categorical propositions of either **A** (All *Ss* are *Ps*) or **E** (No *Ss* are *Ps*) type. Venn diagrams respectively show the relation of S-class and P-class in terms of inclusion and exclusion. Hence, **A** and **E** type propositions, respectively, correspond to analytic and synthetic ones; for if *S* is part of *P* their union is equal to P ($S \subseteq P = S \cup P = P$), whereas if the intersection of *S* and *P* is empty ($S \cap P = \emptyset$) their union is equal to their symmetric difference ($S \emptyset P = (S \cup P) \setminus \emptyset$) and is therefore included in a third class ($Q \subseteq (S \cup P)$). All propositions require some evidence to establish their truth-values. Analytic propositions are true by definition since the S-class is already part of the P-class, while synthetic propositions need something else since their composition is claimed rather than showed. In this latter case, Kant appeals to intuitions.

Geometrical passages are hardly reducible to mere identities in terms of analytic propositions. We "can analyze and clarify the concept of a figure enclosed by three straight lines, and possessing three angles", but we "can never arrive at any property not already contained in these concepts" (A 716/B 744). Consistently, mathematical propositions like those of Euclid's *Elements* are properly explained by synthesis rather than analysis. The former is the only way to add a new property to a concept.

Hence, mathematical concepts are synthetic propositions which carry non-empirical evidence. Kant's account of concepts is not original, however. Basically, it amounts to a variation of the classic bundle theory of Hume. Accordingly, any concept is a collection of properties or, more precisely, a subsumption of properties under a common class. Kant puts a strong emphasis on the function of synthesis which is a priori assumed in any concept formation; in this sense, concepts are a priori synthetic unity of some manifold (A 78–9/B 104–105). In fact, "concepts rest on functions" which are meant as "the unity of the act of bringing various representations under one common representation" (A 68/B 93). The distinguishing factor of mathematical concepts thus pertains to the manifold of representations and not to their synthesis. The latter is a logical operation which mathematical propositions share with all synthetic propositions whatsoever.

Kant has previously argued (A 19–21/B 33–35) that any representation of objects is ultimately an immediate and individual intuition. Hence, to "pass beyond [a concept] to properties which are not contained in this concept, but yet belong to it . . .is impossible unless I determine my object in accordance with the conditions either of empirical or of pure intuition" (A 718/B 746). Therefore, the nature of intuitions eventually decides all synthetic propositions, geometrical inferences included. In fact, a synthetic claim about the world is composed of empirical representations, and its truth-value is ultimately justified by experience, e.g. "the apple is green" (*Ga*) is true or false as long as it can be observed. As true the relation of S-class and P-class claims a composition which my perceptions corroborate. If the claim is not about the experiential world, then other evidence is required.

Hence, mathematical propositions are syntheses of non-empirical intuitions. Kant does not say how they are supported by evidence, but he gives precise indications regarding how they are to be intended. From the latter we can easily derive the former. In what follows I'll try to characterize non-empirical intuitions accordingly.

## Intuitions and mathematical concepts

Kant primarily addresses intuitions in the *Transcendental Aesthetic*. He thinks of them as conditions for everything we may perceive or, more generally, represent (A 19–20/B 33–34). Perceiving and representing overlap only empirically since representations can formally be viewed as entirely pure, like in the case of the forms of intuitions (space and time). In the *Transcendental Doctrine of Method* he further argues that non-empirical intuitions are employed in mathematical constructions (A 713/B 741). The two parts are clearly connected: the former accommodates a metaphysical distinction of intuitions which fits into the mathematical account of the latter. If intuitions were only empirical, mathematics would be consistently abstracted from experience and thus derived via inductive reasoning. Hence, it would be hardly necessary or universally valid.

Such abstractionism, strongly defended by Mill, has been unanimously rejected by the Neo-Kantians, especially by Cassirer. In his *Substance and Function* (1910), he systemically criticizes "the process of comparing things and of grouping them together according to similar properties" since it is still "placed upon thing-concepts" rather than "relation-concepts." This sensualistic approach which also characterizes Husserl's *Philosophy of Arithmetic* (1891) ultimately fails to make sense of mathematical constructions.

In order to avoid that abstractionism, Kant explicitly refers mathematical constructions to *non-empirical* intuitions. He argues that constructing in intuition amounts to an exhibition of concepts *in concreto* – something which involves non-empirical intuitions in a significant way.

For the construction of a concept we therefore need a *non-empirical* intuition. The latter must, as intuition, be a *single* object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. Thus I construct a triangle by representing the object which corresponds to this concept ... The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality (A 713–714/B 741–742).

Non-empirical intuitions behave like individual representations which stand for all other representations; they combine individuality and universality by instantiating the latter under the former. In his First Critique, Kant has thoroughly argued for that; like at the beginning of the *Analytic of Principles* when he says, "transcendental philosophy has the peculiarity that besides the rule ..., which is given in the pure concept of understanding, it can also specify a priori the instance to which the rule is to be applied" (A 135/B 174). Non-empirical intuitions exhibit a priori the concept in a concrete instance. As Kant says, "in mathematics ... the concepts of reason must be forthwith exhibited *in concreto* in pure intuition" (A 711/B 739), therefore "to *construct* a concept means to exhibit *a priori* the intuition which corresponds to it" (A 713/B 741).

The instantiation of universality sharply distinguishes the mathematical concepts from the philosophical ones, and shows them to be an alternative way of conceptualizing. As Kant puts it, "philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance, though still always *a priori* and by means of reason" (A 714/B 742). That corresponds to, "philosophical knowledge ... has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider the universal *in concreto* (in the single intuition) and yet at the same time through pure *a priori* representation" (A 734–735/B 762–763). Non-empirical intuitions clearly realize the idea that a single object or individuality may stand for a manifold of objects or universality, which is exactly the idea of a free variable (*x, y, z*), for instance in first-order logic. Thus, an intuition is intended "as containing an infinite number of representations *within* itself", while a concept is thought of "as a representation which is contained in an infinite number of different possible representations (as their common character), and which therefore contains these *under* itself"; "for all the parts of space coexist *ad infinitum* . . . the original representation of space is an *a priori* intuition, not a concept" (A 25/B 40).

Hintikka explains this perfectly by noting that Kant's mathematical method is "based on the use of constructions" which consists in "introducing particular representatives of general concepts and carrying out arguments in terms of such particular representatives, arguments which cannot be carried out by means of general concepts."[2] Similarly, Parsons argues that "the reasoning involving constructive operations" is carried on "as reasoning with *singular terms,*" and "Kant clearly understood this reasoning as involving singular representations."[3] "Free variables, and terms containing them, have the property that Kant requires of an intuition constructing a concept, in that they are singular and yet also "express universal validity" in the role they play in arguing for general conclusions." The same position is held by Brittan,[4] Friedman,[5] and Thompson.[6].

If we assume that a concept is a collection of properties (individual representations) and each of them is variable or constant (non-empirical or empirical intuition, respectively), then Kant is saying that any philosophical concept unifies individual constants which are derived a posteriori, whereas any mathematical concept unifies individual variables (like a set containing sets) which are available a priori.

The logical character of intuitions also makes sense of the semantic problem, namely "concepts which relate a priori to objects" (A 57/B 81). Since any statement containing individual constants, such as "***F***a," allows for an existential generalization such as "(Øx)***F***x," for any object of experience there is something a priori which allows for its conceptualization.

[2] J. Hintikka, "Kant on the Mathematical Method," *The Monist* 51/3 (1976), 352–375..

[3] C. Parsons, "The Transcendental Aesthetic" in P. Guyer (ed.), *The Cambridge Companion to Kant* (Cambridge University Press, 1992), 62–100; here 78. Parsons also points out that the algebraist's "manipulating symbols according to certain rules [requires] analogous intuitive representation of his concept", and that "the symbolic construction is essentially a construction with *symbols* as objects of intuition." See C. Parsons, "Kant's Philosophy of Arithmetic," in C. J. Posy (ed.), *Kant's Philosophy of Mathematics* (Dordrecht: Kluwer Academic Publishers, 1992), 43–79; here 65.

[4] Already Brittan noticed the difference between intuitions seen as variable or as constant; however he saw them as two ways of constructing in intuitions rather than as pure and empirical intuitions, as I myself do. See G. Brittan, "Algebra and Intuition," in C. J. Posy (ed.), *Kant's Philosophy of Mathematics* (Dordrecht: Kluwer Academic Publishers, 1992), 315–339.

[5] M. Friedman, "Kant on Concepts and Intuitions in the Mathematical Sciences," *Synthese* 84 (1990), 213–257.

[6] M. Thompson, "Singular Terms and Intuitions in Kant's Epistemology," in C. J. Posy (ed.), *Kant's Philosophy of Mathematics* (Dordrecht: Kluwer Academic Publishers, 1992), 81–107.

Otherwise put, in order to represent something as *X,* the representation of *X* must be presupposed (A 250); as Rickert suggests, anything we perceive implies something logical as a priori structure. Hence, the logical-transcendental conditions concerning the possibility of objects of experience ultimately rest on such non-empirical intuitions which behave like free variables. Along with the logical-analytical conditions (concerning the non-contradictoriness of concepts) they make definite sense of Kant's epistemological paradigm as discussed by Parrini.[7]

## Quantifying over intuitions

The construction in intuition discussed above gets further specified as Kant distinguishes between ostensive and symbolic constructions. Respectively, they represent the geometrical and the mathematical way to employ non-empirical intuitions in a logical fashion; namely, as individuals which stand for universals (and not universals which subsume individuals). Both ways significantly involve quantification.

Accordingly, construction in intuitions evolves into construction of magnitudes. This looks quite consistent with the nature of such intuitions. In modern logic binding a variable which ranges over a domain of discourse is called *quantification*. Free variables are intended to be quantified. Consequently, non-empirical intuitions are named as quanta, namely as quantified properties.

But mathematics does not only construct magnitudes (*quanta*) as in geometry; it also constructs magnitude as such (*quantitas*), as in algebra. In this it abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude (A 717/B 745).

Quanta represent the building-blocks of ostensive and symbolic constructions, which are therefore differently quantified. Kant proceeds from a less abstract to a more abstract idea of quantification[8]: "quanta . . . can be

[7] P. Parrini, "Kant's Theory of Knowledge: Truth, Form, Matter," in P. Parrini (ed.), *Kant and Contemporary Epistemology* (Dordrecht, Boston, and London: Kluwer Academic Publishers, 1994), 195–230.

[8] Brittan bestows priority to the arithmetical way of quantifying. While Shabel maintains that symbolic constructions are ultimately species of the ostensive ones: "such as the variable 'x' can be used to represent concretely constructible entity, such as a line segment . . ., the variable symbolically constructs the concrete object by symbolizing the ostensive construction of that object" (Shabel, "Kant's Philosophy of Mathematics," 101). She correctly offers a genealogical suggestion, for the argument derives from examining Euclid. However, logically speaking if *x* stands for everything, including *y* which stands for geometrical figures, then *y* may replace *x* iff *x* is such a figure: if (*y* Ø *x*), then (*y* Ø *x*) but not (*x* Ø *y*).

exhibited *a priori* in intuition, that is, constructed, either in respect of the quality (figure) of the quanta, or through number in their quantity only (the mere synthesis of the homogeneous manifold)" (A 720/B 748). And he finally attributes logical homogeneity to the latter in order to accommodate mereological combinations.

In the metaphysical lectures on Leibniz's principle of indiscernibles (Kant 2001: 29:839), that homogeneity emerges from the lack of qualitative (or specific) difference; this latter connotes a *compositum* with heterogeneity instead.[9] The notion of quantity is primarily, although not exclusively, characterized by that lack. In fact, it also requires a combination of parts, which accordingly are viewed as logically homogeneous. Thus quantity has everything to do with the homogeneity of parts and the way in which they are combined together (*zusammengesetzt*), namely synthesized. Kant's leading idea is that combining parts (the manifold) which are homogeneous leads to magnitudes.

Like Euclid, Kant holds that magnitudes are a combination of parts that are homogeneous. Such combining is to be understood in terms of proportions (ratios). In fact, it is not enough just to belong to the same genus or kind, as the word 'homo-geneous' suggests; the parts must also stay in proportion, namely they must be either bigger or smaller or equal. Any proportion entails homogeneity, but homogeneity does not entail proportion. In order to be in proportion (bigger, smaller, equal) two parts must be conceived as inside of one another, that is, as part and whole. Kant makes sense of it by saying, "A > than B if a part of A=B; in contrast A < B, if A is equal to a part of B" (28:506, late 1780s) or "something is larger than the other if the latter is only equal to a part of the former" (28:561).

This explains quantity in terms of both part-whole relations and homogeneity in a way which is fully consistent with the above definition of "synthesis of the homogeneous manifold". Not just an homogeneity with respect to a more comprehensive concept, but a *strict* homogeneity, namely that which is restricted to proportions or ratios. As Sutherland points out, "homogeneous magnitudes stand in comparative size relations,"[10] because "one magnitude will be larger than another as long as a part of it is equal in

[9] *Quantum* and *compositum* are magnitudes containing a plurality, but only the former accepts homogeneous parts. the latter allows for "an aggregate heterogeneous parts", while the former accepts only homogeneous parts: "[a] composite differs from quantum, and the many would in that case be able to be a variety, every quantum contains a multitude but not every multitude is a quantum; rather [it is one] only when the parts are homogeneous" (Kant 1997: 29:991).

[10] D. Sutherland, "Kant on Arithmetic, Algebra, and the Theory of Proportions," *Journal of the History of Philosophy* 44/4 (2006), 533–558; here 538.

size to the other"; something "which in principle allows the ordering of any two magnitudes of the same kind according to their size", and this by means of their proportions (in symbols: >, <, =) which behave like "invariant under all equimultiple compositions."[11]

In the *Aesthetic*, the pure representations have been already characterized as intuitions of *single wholes*, and these wholes as purely representing *infinite magnitudes* (A 24–5/B 39–40 and A 31–2/B 47–8). For we represent things as placed in different spaces and as related simultaneously or successively; things can only be thought *in* intuitions. Hence, intuitions have something to do with places, and when the same places do not contain anything (existential or empirical) they are still something, namely (ideal or pure) placeholders. It is quite clear that non-empirical intuitions represent wholes containing parts, and that each of these parts can be replaced by anything empirical since its representation is intended as free variable.

## Numbers

Kant primarily discusses numbers in the *Schematism* of his First Critique and in his *Lectures on Metaphysics* $L_2$. In what follows I'll try to make sense of their main thesis, namely that numbers are homogeneous parts combined in succession. To accomplish that I'll explain two intertwined notions, the schema of magnitude and the addition of discrete quanta. Both rely on the logical nature of non-empirical intuitions.

"The subsumption of [objective] intuitions under pure concepts" (A 138/B 177) follows certain rules, which are called by Kant schemata. He restricts such intuitions to time only, and he holds that "the schemata are . . . *a priori* determinations of time in accordance with rules" (A 145/B 187) – a quite misleading move, unless it is properly recognized that spatial intuitions are thoroughly presupposed in any schematization. Despite that, Kant is quite consistent in deriving temporal rules for each group of categories.

What is a schema, exactly? Kant contrasts schemata with images. An image instantiates concepts, e.g. "if five points be set alongside one another, thus . . . have an image of the number five"; whereas a schema generalizes concepts, e.g. "a number in general, whether it be five or a hundred, . . . is rather the representation of a method whereby a multiplicity . . . may be represented in an image in conformity with a certain concept, than the image itself" (A 140/B 179). A schema represents a higher order of universality in contrast to (and in connection with) "some specific universal concept"

[11] Sutherland, "Kant's Philosophy of Mathematics and the Greek Mathematical Traditions," 181, 171.

(A 141/B 180). Kant's arguments implicitly reply to Locke's fictionalism regarding the nature of general ideas.[12]

Indeed it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere (A 141/B 180).

If it is not an image, what is a number in general? Although he definitely restricts his analysis to natural numbers, Kant is looking for a higher order concept whose universality ranges over all numbers – something like the concept of triangle, whose validity regards all triangles.

> . . . the pure *schema* of magnitude (*quantitatis*), as a concept of the understanding, is *number*, a representation which comprises the successive addition of homogeneous units. Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition. (A 142–3/B 182)

My best guess is that the concept of number is precisely a concept that does not collect things but intuitions which stand for things (namely pure intuitions), and that collects them increasingly (namely in addition) by means of some quantification. If I'm right, then numerical collections (numbers) are additions of quantified non-empirical intuitions. Kant's philosophy of mathematics stands or falls on this.

As Rickert correctly points out, additions are not simple conjunctions. While an apple added to another makes two apples, a number added to another does not make two but one, increased number. However, succession alone is not enough to make sense of addition either. An event may follow another without ending up in any union, whereas any number is comprised in the next higher one. Thinking of pure intuitions as placeholders may help to understand their addition. If pure intuitions behave like free variables which stand for any constant, additions may regard logical places instead of empirical things; and the succession in time may be only my way to represent these places. If that is true, then addition may be something which turns the spatial intuitions which temporally follow one another "{Ø}, {Ø}, {Ø}" into their unification in extension "{{{Ø}}}"; however, the number three follows the number one and two because it mereologically includes them and not

[12] E. Carson, "Locke and Kant on Mathematical Knowledge," in E. Carson and R. Huber (eds.), *Intuition and the Axiomatic Method* (Dordrecht: Kluwer Academic Publishers, 2006), 3–20.

simply because it comes after them in conjunction. I believe that Kant shares Benacerraf's Ernie theorem,[13] "for any two numbers, *x* and *y, x* is less than *y* if and only if *x* belongs to *y* and *x* is a proper subset of *y*" vs Johnny's "given two numbers, *x* and *y, x* belongs to *y* if and only if *y* is the successor of *x*" (54); and that he could number three in set-theoretical way: $\mathbf{0}=\varnothing$, $\mathbf{1}=\{0\}=\{\varnothing\}$, $\mathbf{2}=\{0, 1\}=\{\varnothing,\{\varnothing\}\}$, $\mathbf{3} =\{0, 1, 2\}=\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$. For a subset can easily be seen as a part, as indeed Lewis sees it.[14] This certainly deserves a further analysis, involving Frege and Russell along with the relations between set-theory and mereology.

I believe that Kant's mereological characterization of numbers suggests this conclusion, which makes also sense of his above definition. For what characterizes a number is that it comprises the homogeneous units by means of successive addition. The central point seems to me precisely the addition, which Kant clearly intends in mereological terms. The addition of units is carried out by means of comprising parts, and the succession (the time-series, A 145/B 184) merely derives from such comprising parts which enlarges wholes. I also believe that Kant is closer to Cantor and Frege than we use to think.[15]

My evidence relies on *Metaphysics* $L_2$ (28:560–1, 1790–1791), where Kant discusses discrete quanta (another name for number). He holds that "each quantum is as a multitude consisting of homogeneous parts" and that, as such, "each quantum can be increased or decreased." This goes through combining its parts, "the parts that, connected with each other, make a number concept". In this mereological connection "something is larger than the other if the latter is only equal to a part of the former" in fact, "for something to alter into a larger is to increase, and for something to alter into a smaller is to decrease." As Kant says:

> A quantum continuous in itself is one in which the number of parts is indeterminate; a quantum discrete in itself is one in which the number of parts is arbitrarily determined by us. Discrete quantum is therefore called number. Through number we represent each quantum as discrete. (28:561, 1790–1)

The former represents an undetermined multitude (or manifold) of parts because "it does not consist of individual parts." While the latter is made of

[13] P. Benacerraf, "What Numbers Could Not Be" (1965) in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics* (Cambridge University Press, 1983), 272–294.

[14] D. Lewis, *Parts of Classes* (Oxford: Blackwell, 1991).

[15] See C. Parsons, "Arithmetic and the Categories," in C. J. Posy (ed.), *Kant's Philosophy of Mathematics* (Dordrecht: Kluwer Academic Publishers, 1992), 135–158.

partitions, namely "assignable parts." It is noteworthy that space and time are here defined as "continuous quanta" because they contain all assignable parts (as undetermined, though), behaving like a common field in which quanta get differentiated. This field seems to be already predisposed to being quantified by arbitrary partitions. The difference between discrete and continuous quanta rests on the possibility of aggregating parts into wholes. A number is clearly identified with the quantum whose assignable parts are not only homogeneous and individual, but also increased and decreased via aggregation.[16]

This latter says that a number is ultimately an extensive magnitude. Kant calls extensive those magnitudes which are composed by homogeneous parts we can distinguish (28:562), because they are assignable.

> All magnitudes can be considered in two ways: either extensively or intensively. There are objects in which we do distinguish no multitude of homogeneous parts; this is intensive magnitude . . . The objects in which we distinguish a multitude of homogeneous parts have extensive magnitude. The intensive magnitude is the magnitude of the degree, and the extensive magnitude is the magnitude of the aggregate. (28:562)

Differently put, "A magnitude [is] extensive when the representation of the parts makes possible, and therefore necessarily precedes, the representation of the whole" (A 162/B 203). On the contrary, when the composition (*Zusammensetzung*) of parts is a coalition (*Koalition*) rather than an aggregate (*Aggregation*), the magnitude which derives from the mathematical synthesis of the homogeneous is intensive (B 202). For it does not allow for any mereological distinction.[17] While in order to be an aggregate of parts a magnitude must be apprehended in an extensive fashion, namely "through successive synthesis of part to part." This is possible only if all intuitions are from the very beginning conceived in terms of extensive magnitudes (B 202), namely "as aggregates, as complexes of previously given parts" (A 163/B 204).

[16] Schultz says: "1. From several given homogeneous quanta, to generate the concept of one quantum *by their successive connection*, i.e. to transform them into *one whole*. 2. To *increase* and to *diminish* any given quantum *by as much as one wants*, that is, to *infinity*." J. Schultz, *Prüfung der kantischen Kritik der reinen Vernunft* (Königsberg: Hartung, 1789), vol. 1, 221. It is noteworthy that Rickert shares the purely logical argument of Leibniz which grounds arithmetical identity on the associativity and commutativity of addition. The same argument that is employed by Kant's pupil Schultz about progressing from 7 to 12 by successive additions of 1.

[17] Also a continuous quantum can be increased, and that not by means of a mereological addition of parts rather by means of altering the value of its degree; since an intensive magnitudes represent a degree of intuitions (A 166–76/B 207–18).

## Rickert on logical identity

We have seen above how homogeneity and the part–whole relation pertain to Kant's notion of numbers in a crucial way. In his *The One, the Unity and the Number One* (1924a) Rickert offers an interesting variation of that notion by diversifying the logical character of the intuitions. Roughly put, homogenous parts are intended to represent the logical core of numbers, while their mereological compositions are developed in mathematical series. The former are ultimately based upon spatial places, the latter upon temporal series. In what follows, I'll present the arguments in support of these conclusions, starting with homogeneity.

Rickert begins with a metaphysical premise. Logic and mathematics share the same kind of objects, namely the unreal. They both dismiss psychophysical reality, holding that "a number or a line are not real as the paper on which they are drawn or the mental act by which they are perceived" (Rickert 1924a: 3). Yet their similarity ends here. Even if their areas mainly overlap, it may be possible to identify parts of them which are not in common. If it is feasible to do, these parts along with their characteristics could represent "that which is mathematical" (*das Mathematische*) on the one hand, and "that which is truly logical" (*das reine Logische*) on the other.

What is logical? Rickert's first concern is to dismiss any psychological answer in the following way. Anything we may think is an object of thought, and any object of thought is logically something. There is no such thing as thinking in general, which may be considered as empty and independent of any object. We cannot think anything without thinking of something at the same time. However, whereas thinking is an act of my thought, what is thought by such act looks independent of it. Rickert holds that we can sort out act and object from anything we think, and then we can separately question the object of thought without involving the act through which we necessarily think of it. Therefore, the act of thinking and the something which is thought by means of that act can be treated separately.

Rickert accomplishes this goal by carrying out a semantic examination of objects. According to it, any object of thought gets labeled as 'something' and, therefore, gets referred to as a composition of form and content. On the one hand, the word 'something' behaves like a rigid designator à la Kripke so to speak, for it always means that composition. On the other, it can designate anything specific as well, hence it turns out to be non-rigid. Finally, he focuses on several definitions of "something."

(a) In so far as there is a content (whatsoever), there is also a "unifying form of that content." What really matters for Rickert is that "form and content" represent irreducible elements which compose any "something" as its parts or moments. As separated, each of them was not yet something; for in that case, it would rather become itself an object of thought, and as such it would be "something" and thus requiring "form and content" over again. The form itself or the content itself cannot be thought as more elementary objects because any object of thought is a composition of them.

(b) In other terms, form and content represent two logical moments which are sharply different, like "the one and the other" [*das Eine und das Andere*]. He notices that paradoxically the identity of something is connoted by the alterity (otherness) of such moments. Since each of them always remains in relation to the other they form a relation of relatives (*relatio relatorum*). Therefore the logical object results in nothing simple but in a plurality of moments.

The same point is later clarified in *Predicative Logic and The Problem of Ontology* (Rickert 1930), where Rickert defends the idea that the identity of something is ultimately the unity of "the one and the other," for they stand for the relation (*Verbindung*) of subject and predicate, in which alone "the logical meaning is found out in a proposition which relates a grammatical subject with a grammatical predicate, or in short: all true and reasonable propositions are linguistic predications" (Rickert 1930: 30). In this sense, any proposition is the logical, linguistic synthesis of the one and the other. A conclusion otherwise reached in *The Main Problems of Philosophy* (1934), as Rickert says: "the empty form of the subject synthesizes objects without being in turn objectified" for its identity consists in the logical function of objective unification (Rickert 1934: 115–119).

It follows that the ultimate thing of thought or "the most elementary object we may think of" is, logically speaking, a manifold, i.e., not one but two logical places or moments, which are called "pre-objects or pre-objective elements of any object." They represent two constitutive parts of any object of thought, namely of any something in general.

This latter is also addressed as the content which is united by some form. What matters in this case is that they both are strictly formal, namely logical. As the logical content is filled out by anything alogical, the object in general turns particular or empirical. Accordingly, Rickert distinguishes the formal content (*Gehalt*) from the material content (*Inhalt* or *Beschaffenheit*) – the former being constitutive therefore necessary, the latter being only contingent. In this sense, it is always the content that which specifies the kind of object, for the material content that fills the (formal) content is always alogical. It can only be perceived and never thought.

The form alone or the content alone do represent objects which may be thought logically. This is correct and must be highlighted. However it is misleading to believe that they at the same time represent something logically more elementary or that they represent that which is logically ultimate and simplest. On the contrary, that which is theoretically ultimate and simplest has been already obtained by means of the concept of formed content or unity of form and content. The ultimate thing which can be thought as object is never a single thing; such ultimate thing does not designate a singularity but as soon as it is conceptually thought it designates a manifold of elements, which as numbered designates a plurality (Rickert 1924a: 14).

(c) In Rickert's view, form and content replace subject and predicate. He explicitly argues for that in *Predicative Logic* (1930). In his main work *The Object of Knowledge* (1928), he shows how to justify this switch. The argument runs as follows. As we say, for instance, that "the leaf is real" we apply the form of reality (*Wirklichkeit*) to a content that is accordingly real (*wirklich*); however this is not all. We also assume that there is a form in general which is empty and which can be filled with a content; the "leaf" is a real object, that is something that bears "the immanent, real object that is filled in the content" by means of the form of reality. It suggests that form and content of something ultimately mean its subject-predicate relation, and that both are a priori and formal components of any object that we can address as something.

(d) Another name for "something in general" is "one and the same." With this definition Rickert wants to stress that the form of one is what endows the content with unity, and provides something with identity. Such identity connotes a formal property of any object. Rickert expresses this idea in different formulas like "the content in general as unified by the form of identity," or "any content is identified by the form of unity" (Rickert 1924a: 45).

He also introduces a subtle distinction between "what" (whatsoever) and "something," identifying the former with the quality (materiality, *Beschaffenheit*) of the latter; the quality of something is determined by its content. In this sense, form and content are for Rickert replaceable with identity (quantity) and alterity (quality). And from that he criticizes Hegel's dialectical thinking for having based the moment of the antithesis upon negation. In Rickert's eyes it should be more adequately conceived in terms of hetero-thesis based upon alterity, not just a formal negation but a completion. The negation follows alterity, which is not the mere negation of identity but a positive, primitive moment that cannot be simply deduced from the other. Consequently, he replaces tautology with heterology.

In conclusion, Rickert's notion of logical objects describes a triadic construction made of two moments and their union, namely "the one$^{(1)}$ and$^{(3)}$ the other$^{(2)}$." It reproduces the formal synthesis of subject and predicate, which defines the identity of any type of objects, numbers included.

## What numbers could not be

Rickert defends the following thesis. A number is ultimately a structure which is not understandable in merely logical terms, although it is mathematically elementary. This is due to the fact that its concept includes alogical components which prevent it from being purely logical. Consistently, any logical foundation of mathematics is rejected as far as logic is conceived as restricted to the above unity of *S*-term and *P*-term (form and content).

In order to support his conclusions, Rickert introduces a few arguments regarding the non-commutativity of subject and predicate, the irreducibility of identity to equality, and the nature of the numerical series. Let's call this part "what numbers could not be" and let's see how it leads to Rickert's account of numbers.

(a) "Subject and predicate" are not two numbers yet, and their unity does not mean the number two. Although their logical elements numerically designate a plurality, the semantic reference does not mirror the designator. In fact, counting those elements presupposes rather than derives numbers; it looks like numbering anything else. However, let's suppose the opposite to derive a contradiction like in an indirect proof. Thus, if "subject and predicate" meant the same as "1+1" and their "unity" the same as "2," then the commutative law should rule over them like it does over any numerical addition. However, while the two numbers can be switched without any loss, the same cannot happen with that logical unity.

A clarifying example can be drawn from predicative set-theory. The unity subject–predicate specifies the former by means of the latter; it can be read as *x* possessing the property *P*, and upon the propositional function $P(x)$ a set can be built, $S = \{x : P(x)\}$. Rickert argues that *x* and *P* (respectively subject and predicate) do not merely fill the logical places of the set, places which were still remaining after *x* and *P* were removed. On the contrary, they are those places, namely specific building-blocks of any set. As placeholders in any set-building form, they do not have anything in common. Consequently, there is no possibility of switching them.

(b) 'Subject and predicate' are not equal. They identify something by means of their unity but do not allow for any equality among them. Otherwise put, any object is characterized as 'one and the same' by means of the logical relation of its *S*-term with its *P*-term. Far from being equal they are irreducibly different. This is another reason why we cannot count them as numbers; in fact, there is no homogeneity between them which may lead to an addition. Rickert holds that logical identities exclude any duplicate in such a way that we cannot add anything identical to anything.

He draws upon medieval theories sharing the same argument, like those of Meister Eckhardt, who states *Gleichheit steht in Unterschied* (equality rests on difference), or Thomas Aquinas' formula *aequalitas diversorum est* (equality pertains to differences). Accordingly, he even draws the conclusion that there is no adequate semantic expression for identity. Even the classic formula, "A=A" is misleading because it shows two "As" instead of one (and the same). As Friedman nicely puts it, "Rickert still identifies formal logic with the traditional subject-predicate logic, which is indeed confined to relations of genus and species and thus to the purely symmetrical relations of identity and difference."[18]

(c) Finally, the arrangement of "subject and predicate" does not display any mathematical series. Their logical unity lacks any temporal sequence which may lead to a numeration. This is mainly due to the different meaning which is borne by the mathematical sign for addition (*plus*, +) and the logical connective for union (*and*, •). Rickert says that the logical unity connects its elements without any fusion, for the logical elements unified remain separate. It is challenging to express this in current notation.

In my view, it seems his aim is just to exclude any possibility of deriving a series from the disposition of their conjunction: for a series has to do with homogeneous parts, namely a repetition in time and space where a succession can literally take place. In this sense it seems quite consistent to stress the role of mediation played by their union. *S*-term and *P*-term are logical

[18] M. Friedman, *A Parting of the Ways: Carnap, Cassirer, and Heidegger* (Chicago: Open Court, 2000).

places and their union, the logical conjunction, however intended (*and*, •), behaves like a logical medium between heterogeneous parts.

It is noteworthy that switching from "that which is logical" to "that which is mathematical" has significantly to do with the introduction of time in the logical relation of places. It suggests that Rickert sharply differentiates Kant's notion of spatiotemporal intuition in logical, spatial placeholders on the one hand, and their alogical, temporal sequences on the other. Rickert clearly attributes the latter to the subject, "We need to free ourselves from the very idea of a supposed creation of that which is logic by means of the subject, even if such subject is taken as merely formal" (Rickert 1924a: 57). Then he reaffirms the main theme of his philosophy[19] as he had already done in *Two Ways of Epistemology* (1909). The thinking subject can only *recognize* such logical object as something which is *objectively valid* (by itself). Hence, the logical meaning of everything (numbers included) rests in a sort of super-individual, independent sphere. Like a theoretical value which transcends every cognizing subject which is ultimately compelled to recognize it as that which cannot be otherwise. Rickert's philosophy of values is all about this formal obligation of validity, which in the case of numbers turns out to be logical.

In a nice little book on the concept-formation of the sciences, titled *The Theory of Definition* (1888), Rickertaccordingly clarifies the metaphysical nature of mathematical concepts:

> the mathematical concepts are not referred to the sensible, real objects of the natural sciences, from whose vast multiplicity clear properties (*Merkmale*) are first sorted out as essential; but they are valid by virtue of their ideal being, due to which everything is equally essential, namely the difference between essential and non-essential just drops. (Rickert 1888: 43)

If logical objects are not yet numbers, then what are numbers?

## The numerical series

From what numbers cannot be Rickert derives what they are. The logical core of numbers is provided by the subject-predicate relation, whose unity represents the homogenous part of each numerical whole. What remains to

[19] Mathematics works with objects which are quantitatively determined. There are three ontological spheres: logic (that which is valid, or has validity, or simply is without existing), mathematics (ideal existing) and reality (real existing). The validity of logical objects (objects which only own a general content (not a particular or special content filling that general content) disregards any quantification along with all existence. It represents a duty or a norm for the thinking subject who does not have any other choice rather than recognizing its validity.

be discussed are the alogical components of the numbers. They eventually coincide with the thought itself, namely the real, psychical process (act) that takes place in time involving series and quantity. I'll first survey Rickert's argument along with its sources (Dedekind), and then I'll compare both of them with the Neo-Kantians of Marburg.

The alogical components by themselves cannot be numbers, for instance *the acts of thought* which count objects in time are not numbers. But they say something crucial about them. Thinking the same object for several times does not result in different objects but in different acts of thinking one and the same object. "The identity of an object can be placed many times, although to each of these times does not correspond a different identity", for there were no two identities but "just two times the same identity"; more precisely, "the same union of form and content (subject-predicate relation) is repeated many times" (Rickert 1924a: 56). This suggests to Rickert that numbers have to do with psychical (empirical) acts which are repeated in time; by means of them the same object is placed over and over again. However, in order to accomplish such repetition a *temporal series* is needed, for the placement is a psychical reality which occurs in a time series (*Reihe*).

In such series we find the same object in different places, something which gathers together that identity (*Ps*) and diversity (different places on a line) required for the mathematical equality. Thus, switching one logical object with another becomes possible, for the series works like an homogeneous medium where one and the same object can be differently placed somewhere else; and in this sense it can finally be seen as equal. In comparison, the logical medium was pretty much heterogenous.

Although that explains equality and commutativity, it does not yet account for numerical addition. In fact, the series as such (the homogeneous medium) just represents a bunch of places filled by one and the same object. Rickert notices that and introduces *quantity* as the last alogical component. He precisely introduces quantity into the formal content of something in general as a sort of special, qualitative content. In this sense, a number is something in general (a logical object) which is qualified by a certain quantity and thus assigned to (placed in) a precise position of any series whatsoever. In other terms, it is a quantum. The quantification turns a bunch of places into an ordered series which by virtue of that looks like an arrow pointing at a progression.

Zijderveld captures Rickert's idea as follows: "As abstract, pure, ideal and non-sensual as numerals are, they are nevertheless the substantial, quantitative objects of mathematics" since "after all numerals do exist."[20] However, I think that Rickert's reasoning is missing a point. Introducing quantity in each place of the series is ineffective unless those places get somehow aggregated (like parts) in a proportional way. As far as Rickert keeps quantitative sizes as components of numbers, he could not dismiss part-whole relations. The entire argument asks for a mereological foundation of numbers in a Kantian sense.

About that, it is interesting to notice how Rickert diverges from Cassirer after their initial, common endorsement of Dedekind's theory of numbers.[21] For Dedekind the whole of arithmetic follows from the act of counting which bears out "the successive creation of the infinite series of positive integers in which each individual is defined by the one immediately preceding." If $a$ and $b$ represent one and the same rational number, then $a=b$ as well as $b=a$. If not, then their difference $a–b$ has either a positive or negative value. "In the former case $a$ is said to be greater than $b$, $b$ less than $a$," in symbols as $a > b$, $b < a$; as "in the latter case $b–a$ has a positive value it follows that $b > a$, $a < b$" (Dedekind 1901: 4–5). The three following laws ground Rickert's notion of numerical series and the entire mathematical account of the Marburg Neo-Kantians.

> **i.** If $a>b$, and $b>c$, then $a>c$. Whenever $a$, $c$ are two different (or unequal) numbers, and $b$ is greater than the one and less than the other, [it geometrically means] $b$ lies between the two numbers $a$, $c$. [That is] If $p$ lies to the right of $q$, and $q$ to the right of $r$, then $p$ lies to the right of $r$, . . . $q$ lies between the points $p$ and $r$. **ii.** If $a$, $c$ are two different numbers, there are infinitely many different numbers lying between $a$, $c$. [That is] If $p$, $r$ are two different points, then there always exist infinitely many points that lie between $p$ and $r$. **iii.** If $a$ is any definite number, then all numbers of the system $R$ fall into two classes, $A_1$ and $A_2$, each of which contains infinitely many individuals; . . . $A_1$ comprises all numbers $a_1$ that are $<a$, . . . $A_2$ comprises all numbers $a_2$ that are $>a$ . . . every number of . . . $A_1$ is less than every number of . . . $A_2$. [That is] If $p$ is a definite point in $L$, then all points in $L$ fall into two classes, $P_1$, $P_2$, each of which contains infinitely many individuals; . . . $P_1$ contains all the points $p_1$, that lie to the left of $p$, and . . . $P_2$ contains all the points $p_2$ that lie to the right of $p$; the point $p$ itself may be assigned at pleasure to the first or second class. (Dedekind 1901: 6–7)

Dedekind shows that the "properties of rational numbers recall the corresponding relations of position of the points of a straight line $L$"

[20] A. Zijderveld, *Rickert's Relevance: The Ontological Nature and Epistemological Functions of Values* (Leiden: Brill, 2006), 156.

[21] R. Dedekind, "Continuity and Irrational Numbers," in *Essays on the Theory of Numbers* (Chicago: Open Court, 1901), 1–27.

(Dedekind 1901: 6), where the arithmetic difference assigns each number to a different place along the series. According to Dedekind's relational structure, "numbers are simply places within such a series or progression", and their concept is properly defined as relational for it is "entirely exhausted by the formal properties of a particular kind of relational structure."[22]

This is precisely the notion of numerical series loosely discussed by Rickert and endorsed by Cassirer. In his *Substance and Function* (1923) Cassirer argues that natural numbers are *entirely* given by their position in the progression of ordinal numbers. They behave like objects whose system is ruled by a logic of relations (the concepts of function), objects that are thus "resolved into a web of relations" (Cassirer 1927: 31–92). Given a certain law of progression, "to every member there belongs an immediate successor with which it is connected by an unambiguous transitive and asymmetrical relation". Since mathematics "is a system of ideal objects whose whole content is exhausted in their mutual relations, . . . the 'essence' of the numbers is completely expressed in their positions", and they solely derive "from purely logical premises" (Cassirer 1923: 38–39).

"When," says Dedekind in definition, "in the consideration of a simple infinite system N, arranged by the "copying" (*Abbildung*) $\Phi$, we totally abstract from the particular properties of the elements, retain merely their distinctness, and attend only to the relations in which they are placed to each other by the ordering "copying" $\Phi$, then these elements are called the natural numbers or the ordinal numbers or also simply numbers, and the fundamental element 1 is called the fundamental number of the numerical series N" (Cassirer 1923: 38).

Rickert agrees with Cassirer's premises but not with his conclusions. The schools they respectively represent, the Baden and Marburg Schools of Neo-Kantianism, converge at "the relationship between mathematics and the realm of pure logic."[23] In fact, pure logic is certainly included within mathematics but Rickert balks at the converse. Mathematical objects like numbers represents a sort of advanced structures which are not ultimately reducible in the logical terms of predication or in any of its formal variations, set-predicative theory included. Like Rickert, Natorp[24] argues that "arithmetic is grounded on a series of relations among elements (namely the natural numbers ordered by the successor relation) . . . but he adds that the sequence so defined is not identical to or dependent on

[22] Friedman, *A Parting of the Ways*, 29.
[23] Friedman, *A Parting of the Ways*, 29.
[24] Natorp 1910.

temporal sequence."[25] A conclusion that Rickert explicitly rejects along with Natorp's logicism. Natorp holds that the fundamental moments of thought do not concern space and time but the separation and unification of equals in such a way that temporal succession is excluded. Rickert challenges this conception arguing that as far as the objects arranged in serial order are homogeneous their identity is not derived from that order but it is constantly presupposed in the homogeneity which allows for the addition of equals. In contrast to Cassirer's and Natorp's *logical idealism*, Rickert names his position *transcendental empiricism*, thus emphasizing the alogical components of the concept number.

In conclusion, Rickert accounts for numbers in terms of quantitative units placed in a progressive series, by involving logical and alogical components, along with their spatial and temporal properties. The former are provided by the subject-predicate relation, the latter are introduced by a counting subject and Dedekind's relations of places. The nature of numbers is ultimately decided by their addition, namely the fusion of equal (homogeneous) parts which results in the variation of quantity according to a progression. In this sense, Rickert holds that smaller (less than, <) and larger (more than, >) numbers are qualitative inequalities of *quantity,* namely differentiated identities. This brings Rickert's account of numbers closer to Kant's mereology than to the relationism of the Marburg Neo-Kantians.

[25] Jeremy Heis, "Critical philosophy begins at the very point where logistic leaves off: Cassirer's response to Frege and Russell," *Perspectives on Science*, 18/4 (2010), 383–408; here 393.