On Kantian Intuitions

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In regard to the theoretical philosophy of Kant, we can certainly agree on the relevance of intuitions.[1] Each main division of the first *Critique* specifically depends upon their nature and function. Our knowledge of objects begins with intuitions (*Aesthetic*), is limited to intuitions (*Logic*), and is ultimately possible because of intuitions (*Doctrine of Method*). Nonetheless, there is disagreement about how to understand them, even among philosophers of mathematics. For instance, those who derive synthetic truths from axioms usually downplay the role of intuitions in both Kant and mathematics (see Frege, 1893). In opposition to them, Hintikka provides a revolutionary interpretation, somehow anticipated by Beth and developed by Parsons and Friedman (among others). According to them, Kantian intuitions are primarily intended for mathematical inferences, and thus behave like variables (x, y, z) in first-order quantificational logic.

Building on these latter references, I will present and defend the logical interpretation of the Kantian notion of intuition, including its consistency throughout the first *Critique*. My presentation will rely on three premises: ***1)*** Intuitions will, first, be introduced as objective representations. I will explain their nature in relation to the ideality of space and time, from which will emerge their mereological connotations. ***2)***. As references to objects, intuitions will be characterized by the evidence they provide our semantics. Such evidence will initially be restricted to analytic and synthetic claims and, more precisely, to the basic class relations of inclusion and exclusion. ***3)***. I then extend my discussion to synthetic a priori claims by involving intuition-based constructions borrowed from Euclid's geometry. ***4)***. In a forth section I introduce the "Hintikka-Parsons argument", according to which Kantian intuitions qualify as instances of logical variables.

[1] I would like to thank Lois Parkinson Zamora for all her insightful editing, and James Garson for his wonderful comments and critical remarks.

After its presentation I will also defend this logical character of intuitions by appealing to ***5)*** the Kantian notion of quanta, and ***6)*** mathematical induction. In the first case, the notion of intuition will be developed into the notion of quantified partition, upon which Kant grounds the number concept. In the second case, I will show how that role which Kant attributes to intuitions is played in mathematics by variables that allow for inductive proofs.

1. Intuitions

The notion of intuition is primarily discussed in the *Aesthetic*. Intuitions are representations of objects (B41), although such a definition could seem misleading because there are actually no objects without intuitions. In fact, they might be better defined as components of objects insofar as we recognize that objects are constructions which require certain structural components. In this sense, Kant thinks of intuitions as conditions of possibility for objective representations; and, accordingly, he thinks of an object as a collection of representations (properties), each of them corresponding to an intuition[2]: “objects are nothing but mere representations of our sensibility” (A30/B45).

Let’s recall that Kant properly distinguishes between things as ‘they are in themselves’ (at best, noumena) and things as ‘they appear to us’ (phenomena). These are not separate objects but two sides of one and the same object (many readers of Kant stress the difference rather than the relation of the two). Kant does not hold a Platonic view of reality: intuitions do not represent independent objects, strictly speaking, but rather grasp (or receive) something of the things in themselves and represent it objectively; such things in themselves are “the true correlate of sensibility” (A30/B45), and “by that is meant a something = *X*” (A250), which is always presupposed in that which appears.

Intuitions are further classified as empirical and pure representations. Intuitions turn empirical as sensation comes into play. This latter is a posteriori since it derives from an affection: “that intuition which is in relation to the object through sensation, is entitled *empirical*”; in contrast to sensations, intuitions are a priori components of objective representations, whereas representations “in which there is nothing that belongs to sensation” are *pure* (A20/B34). Thus, any object is initially given as a collection of empirical intuitions, but neither the sensation nor the collection pertains to pure intuition. The former (sensation) is provided by

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[2] See Philip Kitcher, 1992.

affections of our sensibility, the latter (collection) by some categories of our thought. Kant holds that the manifold of intuitions comes blindly without synthesis, whose unifying function derives from logical activity. Therefore, the access to pure intuition requires a double abstraction.

> If I take away from the representation of a body that which the understanding thinks in regard to it, substance, force, divisibility, etc., and likewise what belongs to sensation, impenetrability, hardness, colour, etc., something still remains over from this empirical intuition, namely, extension and figure. These belong to pure intuition, which, even without any actual object of the senses or of sensation, exists in the mind *a priori* as a mere form of sensibility. (A20-1/B35)

Since pure intuitions are what remain after abstraction, they are structural components (a priori conditions) of any object, given that no further abstraction from objects is possible. Kant is saying that we cannot represent an object without such components. This impossibility is properly addressed by Allison: "if x can be (or be represented) without A, B, C and their mutual relations, but A, B, C and their mutual relations cannot be (or be represented) without x, then x must be viewed as a condition of the possibility of A, B, C and their mutual relations (or the representation thereof)" (2004: 104). This makes clear Kant's reason for immediately identifying the forms of sensibility with space and time[3]. If an object is a collection of representations, each representation occupies a place in space; if abstracted from anything concrete, the collection itself (synthesis) nonetheless remains along with its ideal places. These ideal spaces are pure intuitions that, accordingly, behave like placeholders – a position I will pursue by referring to statement functions in predicative logic.

Let's recall that a statement function ($\boldsymbol{F}$) is what remains after a quantifier is removed, namely a pattern for statements whose variables are freed. If we look at the rules for removing and introducing quantifiers (which allow for deduction otherwise impossible), Kant, in this case, relies on existential generalization (EG): from En or Ey to $(\exists x)Ex$, namely if $\boldsymbol{F}a$ or $\boldsymbol{F}y$ then $(\exists x)\boldsymbol{F}x$ (e.g., if one or all numbers of A are even, then some number in A is even). In this way, Kant can properly derive bound variables (via EG) from statements made of constants (empirical intuitions), and eventually free them (pure intuitions) and play with them.

[3] A different view is held by Pippin (1982: 55).

Consistently, Kant defends the ideality of space and time.[4] ***a)*** Space and time are constantly presupposed by our intuitions, each of them taking place within the spatiotemporal framework, and ***b)*** because they are still something after everything empirical is removed, they are something ideal. Kant elaborates these matters as follows: ***a)*** In order to represent objects "as outside and alongside one another, and accordingly as not only different but in different places, the representation of space must be presupposed" (A23/B38); and, "Only on the presupposition of time can we represent to ourselves a number of things as existing at one and the same time (simultaneously) or at different times (successively)" (A30/B46). ***b)*** "We can never represent to ourselves the absence of space, though we can quite well think it as empty of objects" (A24/B38); and, "We cannot, in respect of appearances in general, remove time itself, though we can quite well think time as void of appearances" (A31/B46).

Space and time have everything to do with places, that is, their geometrical and topological characteristics. This feature reveals the mereological nature of intuitions[5], which allows for compositions of parts and wholes. With pure intuition, "all the manifold of appearance is intuited in certain relations" (A20/B35), namely determined and ordered in ***a)*** space and ***b)*** time. These relations are further described in mereological terms: ***a)*** diverse spaces are "only parts of one and the same unique space" (A25/B39), like ***b)*** "different times are but parts of one and the same time" (A31/B47). Thanks to the nature of our intuitions, we can even play with these compositions: ***a)*** "space is essentially one; the manifold in it [...] depends solely on [the introduction of] limitations", but "these parts cannot precede the one all-embracing space, as being [...] constituents out of which it can be composed; on the contrary, they can be thought only as *in* it" (A25/B39). Similarly, ***b)*** "every determinate magnitude of time is possible only through limitations of one single time that underlies it" (A32/B48) because the original representation of time is given as unlimited. Space and time behave like the notion of universe in logic,

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[4] I will *not* address the controversial conclusions that Kant draws. He defines space and time as necessary a priori representations that cannot be empirically obtained, whose nature is ***a)*** neither an absolute, self-subsistent entity ***b)*** nor a property of things in themselves, but ***c)*** a subjective condition (of sensibility) that is both prior to all experience and determining the relations of objects in that experience; therefore ***d)*** they are *ideal* before things in themselves but also *real* because they possess objective validity before all appearances. Notoriously, Kant neglects a third alternative, "namely, that though our intuition of space is subjective in origin, space is itself an inherent property of things in themselves" (Kemp Smith, 1918: 113).

[5] See Richard E. Aquila, 1994.

namely like a whole whose parts (or quantities) are represented only through limitations of an infinitude that is ideal and original. The only difference between space and time is that "all parts of space coexist *ad infinitum*" (A25/B40), while "different times are not simultaneous but successive (just as different spaces are not successive but simultaneous)" because "time has only one dimension" (A31/B47).

Let's draw a partial conclusion. In the *Aesthetic*, Kantian intuitions carry two logical connotations. First, they behave like placeholders: since pure intuitions stand for anything empirical whatsoever, they recall the idea of logical variable. Second, the framework of these intuitions is characterized by part-whole relations. And intuitions represent such relationship.

In what follows, I further investigate the logical nature of Kantian intuitions addressing their cognitive function.

2. Analytic and Synthetic Judgments

The relation between intuitions and judgments (and their conceptual terms) is the basis of Kant's strategy to merge empiricism and rationalism, and is, indeed, what his theoretical philosophy is all about. I see two specific reasons why he associates intuitions and judgments. ***a)*** As a collection of representations, each of them corresponding to an intuition, an object (like a whole) always entails a certain unity of its parts. Each representation of an object is called "partial representation" (A32/B48), and the unity (whole) of all partial representations is the object itself. Therefore, intuitions are from the outset subject to some synthesis (category of thought or concept) – let's briefly recall that a category of thought is a kind of synthesis, which is also called a concept. Thus, "concepts rest on functions", and by "function" Kant means "the unity of the act of bringing various representations under one common representation" (A68/B93). Similarly, "all judgments are functions of unity among our representations" (A69/B93), but these latter are *higher* representations, namely concepts that comprise *immediate* representations (i.e., intuitions). ***b)*** As references to objects (A19/B33), intuitions provide evidence for our semantics. Consider the case of "a combination of contradictorily opposed predicates in one and the same object, for instance, the being and the not-being of one and the same thing in one and the same place", a case that rests entirely on intuitions: "Only in time can two contradictorily opposed predicates meet in one and the same object, namely, *one after the other*" (B48-9). Intuitions ultimately justify any kind of judgment, starting with the basic analytic and synthetic ones.

If we stipulate that a proposition is either true or false (but not undecided[6]), then establishing the truth-value requires evidence of some sort. As Ayer (1952: 71-87) puts it, all evidence can be either empirical or analytical, the former relying on experience, the latter deriving from pre-established compositions (of mereological or set-theoretic nature). Sentences like 'the apple is green' (*Ga*) are true or false insofar as the parts and their a posteriori relation can be observed: their truth relies on some correspondence between experience and semantics. Sentences like 'apples are fruit' (x: $Ax \supset Fx$) are true by definition since their parts are a priori related by linguistic convention: their truth relies on the coherence of the part-whole relations.

> In all judgments in which the relation of a subject to the predicate is thought (I take into consideration affirmative judgments only, the subsequent application to negative judgments being easily made), this relation is possible in two different ways. Either the predicate B belongs to the subject A, as something which is (covertly) contained in this concept A; or B lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgment analytic, in the other synthetic. (A6-7/B10)

Kant is saying that all judgments are reducible to categorical propositions of either **A** (All *Ss* are *Ps*) / **E** (No *Ss* are *Ps*), or **I** (Some *Ss* are *Ps*) / **O** (Some *Ss* are not *Ps*) types, the Venn diagrams of which show the relations of *S*-class and *P*-class in terms of inclusion and exclusion. Kant centers the difference among judgments on the distribution of *S*-class – let's recall that a term is distributed if the proposition says something about every member of the class denoted by the term. ***a)*** If distributed, either *S* behaves like a subset of *P*, whose relation with *P* turns into a union equal to *P*: if $S \subseteq P$, $S \cup P = P$; or *S* behaves like a disjoint set, whose relation with *P* turns into an intersection equal to an empty set: if *S* and *P* are disjoint, $S \cap P = \varnothing$. The two relations, respectively, instantiate inclusion and exclusion in universal claims. ***b)*** If *S* is *not* distributed, either *S*'s relation to *P* behaves like an intersection of two sets, which is equal to the set of all elements belonging to both *S* and *P*: that is, $S \cap P = \{x: x \in S \text{ and } x \in P\}$; or *S*'s relation to *P* behaves like the symmetric difference of *S* and *P*, which consists of all elements belonging to *S* or *P* but not to both *S* and *P*: that is, $S \oplus P = (S \cup P) \setminus (S \cap P)$. The two relations, respectively, instantiate inclusion and exclusion in particular claims.

[6] See James W. Garson, 2014.

Analytic and synthetic judgments, respectively, correspond to **A/E** and **I/O** types of claims. Kant simplifies and considers only **A** type, where *S* behaves like a proper subset of *P* (if $S \subset P$, $S \cup P = P$), and **I** type, where the intersection of *S* and *P* is equal to a non-empty subset of *S* and *P* ($S \cap P \subset S$ and $S \cap P \subset P$). In both cases, the justification of the truth-values requires some evidence: ***a)*** analytic judgments rely on the coherent distribution of their parts, ***b)*** synthetic judgments require some sort of correspondence. And in both cases, intuitions provide such evidence: respectively, ***a)*** as parts mereologically related, and ***b)*** as references to objects. Consistently, if intuitions are pure, the evidence supplied is non-empirical.

Given two classes (like *S* and *P*), analysis and synthesis[7] are operations to derive a further part.[8] *P*-class includes *S*-class if a part of *P*-class is equal to *S*-class; asymmetrically, *S*-class and *P*-class are synthesized if they share a common part. In the former case, the part in question is derived (analyzed) from *P*-class; in the latter case, it is produced from the overlapping of *S*-class and *P*-class. Analysis and synthesis are ways to obtain the wanted part within the scope of Kantian judgments, especially of those related to quantity which are explicitly defined by "the relations of inclusion or exclusion between extensions or spheres of concepts" (Longuenesse, 1998: 247).

> As to quantity, judgments are either *universal* or *particular* or *singular*, accordingly as the subject is either *wholly in*cluded in or *ex*cluded from the notion of the predicate or is only *in part in*cluded in or *ex*cluded from it. In the *universal* judgment, the sphere of one concept is wholly enclosed within the sphere of another; in the *particular*, a part of the former is enclosed under the sphere of the other; and in the *singular* judgment, finally, a concept that has no sphere at all is enclosed, merely as part then, under the sphere of another. (Log AA 9:102)

As logical operations, analysis and synthesis require proofs. Deduction and induction seem to satisfy this requirement, since they pertain to proving processes. But both have downsides. Since they are obtained via induction, synthetic claims can only be substantiated in $n-1$ cases without any guarantee of being further substantiated in the n^{th} case. No matter how large *n* is taken to be, there still remains the possibility that our claims will be confuted on some future occasion. "What is derived from experience has only comparative universality, namely, that which is obtained through induction" (A24); therefore experience gives "neither strict universality

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[7] See Michael Kremer, 2006.
[8] See David K. Lewis, 1991.

nor apodeictic certainty" (A31/B47). Thus, whatever is derived from experience looks alike, namely contingent. At best, synthetic claims are valid generalizations, thus reducible to Hume's custom-induced inferences. This is not the case with analytic claims based on deduction, which, however, are ultimately resolved into mere tautologies, namely identities broken up into as many steps as, for instance, the calculus requires. The deduction from one step to another bestows necessity on analytic claims but deprives them of any possibility of yielding new knowledge.

In conclusion, Kantian intuitions clearly have a cognitive function. They justify judgments establishing the truth-values of terms and term-relations. In fact, the inclusion and exclusion of *S*-term and *P*-term are operations that pertain to analytic and synthetic compositions of intuitions.

3. Synthetic A Priori Judgments

Beside the analytic and synthetic,[9] Kant identifies a controversial third kind of judgment, namely the synthetic a priori. In the *Doctrine of Method*,[10] he looks to the Euclidean model. Geometric claims genuinely challenge the analytic-synthetic distinction. On the one hand, they are synthetic, but on the other, they carry analytic connotations like strict universality (necessity). These claims are hardly reducible to mere identities in terms of tautologies: one can analyze and clarify "the concept of a figure enclosed by three straight lines, and possessing three angles", but "she can never arrive at any properties not already contained in these concepts" (A716/B744). Furthermore, to add a new property to a concept is a synthetic rather than an analytic operation. However, all inferences of geometry are carried out with strict and not merely comparative universality, i.e., *Ps* "cannot be otherwise" (A1/B3), that is, it is true in all possible worlds. Therefore, geometric claims are syntheses whose evidence derives not from experience (no abstraction is there involved). If they were empirical, they would be abstracted from particular instances via inductive reasoning, and thus would be neither necessary nor universally valid.

Kant classifies claims like the Euclidean as 'synthetic a priori', and attributes their special nature to intuitions. As he anticipates in the *Aesthetic*, "such judgments" are ultimately "based on intuition" (B73) in a non empirical way. They rather represent 'intuition based constructions' because their truth-value runs from one claim to another "through a chain

[9] See R. Lanier Anderson, 2015.
[10] See Lisa A. Shabel, 2006.

of inferences guided throughout by intuition" (A716/B744).

> Here, then, in pure *a priori* intuitions, space and time, we have one of the factors required for solution of the general problem of transcendental philosophy: *how are synthetic a priori judgments possible?* When in *a priori* judgment we seek to go out beyond the given concept, we come in the *a priori* intuitions upon that which cannot be discovered in the concept but which is certainly found *a priori* in the intuition corresponding to the concept, and can be connected with it synthetically. (B73)

Since "from a mere concept no propositions can be obtained which go beyond the concept" (B41), to "pass beyond [a concept] to properties which are not contained in this concept, but yet belong to it [...] is impossible unless I determine my object in accordance with the conditions either of empirical or of pure intuition" (A718/B746). Therefore, the nature of intuitions eventually decides all kinds of synthetic judgments, including a priori ones. Kant extends the same conclusion to all mathematical claims broadly conceived, since as conceptual syntheses of pure intuitions they also carry non-empirical evidence. But how are these intuitions intended? Kant looks again to the Euclidian "inferences guided throughout by intuition" (A716/B744), and identifies two characteristics.

a) The evidence that replaces experience in geometric claims derives from the exhibition of concepts *in concreto*. Thus, mathematical concepts are somehow exhibited a priori by means of pure intuitions. In his first *Critique*, Kant makes this argument starting from the *Aesthetic*: "You must therefore give yourself an object a priori in intuition, and ground upon this your synthetic proposition" (A48/B65)." At the beginning of the *Analytic of Principles*, he adds: "transcendental philosophy has the peculiarity that besides the rule [...], which is given in the pure concept of understanding, it can also specify a priori the instance to which the rule is to be applied" (A135/B174). In the *Doctrine of Method*, he clarifies: "in mathematics [...] the concepts of reason must be forthwith exhibited *in concreto* in pure intuition" (A711/B739); therefore, "to *construct* a concept means to exhibit *a priori* the intuition which corresponds to it" (A713/B741).

b) The way in which mathematical thinking differs from philosophical analysis tells us more about the nature of intuitions: the latter is confined to concepts; the former pertains to the construction of concepts. According to Kant, "philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance, though still always *a priori* and by means of reason" (A714/B742). Furthermore, "philosophical knowledge [...] has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider the universal *in concreto* (in the single intuition) and yet at the same time through pure *a priori* representation" (A734-5/B762-3).

> For the construction of a concept we therefore need a *non-empirical* intuition. The latter must, as intuition, be a *single* object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. Thus I construct a triangle by representing the object which corresponds to this concept either by imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition – in both cases completely *a priori*, without having borrowed the pattern from any experience. The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. (A713-4/B741-2)

a) To exhibit a concept *in concreto* means ***b)*** to represent something universal (like a concept) in a single instance. Pure intuitions are thus specified as individual instances of universal representations. Therefore, an intuition is "thought as containing an infinite number of representations *within* itself", while a concept is "thought as a representation which is contained in an infinite number of different possible representations (as their common character), and which therefore contains these *under* itself" (A25/B40).

Hence, Kantian intuitions are the answer to the main question of the first *Critique*, namely "How are a priori synthetic judgments possible?" (B 19). Kant's theoretical philosophy stands or falls on this. Let's see how to properly understand these intuitions and their logical nature.

4. The Hintikka-Parsons Argument

Despite a few differences, Hintikka and Parsons offer a joint interpretation of Kantian intuitions that accommodates all of their characteristics within a logical picture. I call it 'the Hintikka-Parsons argument' (HP). HP moves from a premise concerning mathematical claims: since pure intuitions are all about representing the universal in the single instance, they consist in individual representations that stand for other representations. In this sense, individuality and universality are so combined that the latter is instantiated under the former. Therefore, HP concludes, these intuitions clearly realize the idea that a single object (individuality) may stand for a plurality of objects (universality), which is exactly the idea of free variables (*x, y, z*, and their inferences), for instance in first-order quantificational logic[11]. Let's take a closer look.

Recall that, according to Kant, to construct a concept means to exhibit, a priori, an intuition that corresponds to the concept. Hintikka (1992[1]) inquiries about the nature of these intuitions. He starts by detaching them from any mental picture or anything visual: there is, he says, "nothing intuitive about intuitions" (23). He then points to their individuality and the role they play in mathematical constructions: "in mathematics, one is all the time introducing particular representatives of general concepts and carrying out arguments in terms of such particular representatives" (24). In fact, using algebraic symbols is not meant to furnish ourselves with images. Kantian intuitions, like those symbols, are representatives of general concepts. In this case, "if we can assume that the symbols we use in algebra stand for individual numbers, then it becomes trivially true to say that algebra is based on the use of intuitions" (26-7).

> Then we can also understand what Kant had in mind when he called algebraic operations, such as addition, multiplication, and division, constructions. For what happens when we combine in algebra two letters, say *a* and *b*, with a functional sign, be this *f* or *g* or + or • or :, obtaining an expression like *f*(*a*, *b*) or *g*(*a*, *b*) or *a* + *b* or *a* • *b* or *a* : *b*? These expressions, obviously, stand for individual numbers or, more generally, for individual magnitudes, usually for individuals different from those for which *a* and *b* stood for. What has happened, therefore, is that we have introduced a representative for a new individual. And such an introduction of representatives for new individuals, i.e., new intuitions, was just what according to Kant's definition happens when we construct something. The new individuals may be said to represent the concepts 'the sum of *a* and *b*', 'the product of *a* and *b*', etc. (27)

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[11] There is an objection to HP that moves from Kant's examples of synthetic a priori judgments, like "everything which happens has its cause" (A9/B13), and thus subverts HP's main assumption that the first *Critique* is centered on mathematical claims. However, the primacy of mathematical method is systematic throughout the first *Critique*. Not only was the *Doctrine of Method* written prior to the *Aesthetic*, but also the *Prolegomena*, the work that is supposed to explain the arguments of the first *Critique*, explicitly appeals to mathematical claims from the very outset. In regard to synthetic a priori judgments, Kant clarifies that "with the concept of cause I do really go beyond the empirical concept of an event (something happening), yet I do not pass to the intuition which exhibits the concept of cause *in concreto* [...] I therefore proceed merely in accordance with concepts; I cannot proceed by means of the construction of concepts" (A722/B750). This evidence is simply overlooked by critics of HP.

In this context, analytic and synthetic are two methods of ‘carrying out a proof’ or ‘construction.’ They represent active processes of *seeking* and *finding* that cannot pertain to perceptions. Hintikka rejects the idea of reducing Kantian intuitions to such passive perceptions. If it were the case, he claims, Kant’s doctrine would be “hopelessly wrong” because we can know individuals “even in areas where sense-perception is not involved at all, e.g., in dealing with numbers and other abstract entities” (1992^2: 348). Consistently, Hintikka criticizes Parsons’s insistence on the immediacy of intuitions. In this sense, “*a priori* intuitions are not characterized by an especially immediate relation to their objects; they are precisely intuitions used in the absence of their objects” (358). Therefore, he concludes, they can hardly be intended as perceptual.

But, replies Parsons, the symbols implied by conceptual constructions in intuition[12] are perceptible objects. Those constructions need something phenomenological[13] like perceptions (represented by single instances). In this regard, Hintikka fails to account for the sensible side of Kantian intuitions[14]. Furthermore, though immediacy entails singularity, the reverse, says Parsons, need not hold. Therefore, the immediacy criterion is not “simply a corollary of the individuality criterion” (342), as Hintikka mistakenly holds. Despite this, Parsons endorses Hintikka’s logical interpretation and claims: “a picture common to us is of pure intuitions as analogous to *free variables*, with predicates attached to them representing the concepts they construct” (1992^4: 74). In fact, “free variables, and terms containing them, have the property that Kant requires of an intuition constructing a concept, in that they are singular and yet also ‘express universal validity’ in the role they play in arguing for general conclusions” (1992^2: 78). But this is not all they have in common.

Hintikka and Parsons also share the reference to Beth.[15] In the proof that the base angles of an isosceles triangle are equal, Beth was the first to notice that:

> We proceed, as is well known, as a rule as follows: first we consider a particular triangle, say ABC, and suppose that AB=AC; then we show that ∠ABC=∠ACB and have thus proved that the assertion holds in the particular case in question. Then one observes that the proof is correct for an arbitrary triangle, and therefore that the assertion must hold in general (1956-7: 365).

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[12] See Gordon G. Brittan, 1992.
[13] See Carl J. Posy, 2000.
[14] See Manley Thompson, 1992.
[15] See Michael Friedman, 1990.

Parsons reads Beth's argument as a case of universal generalization (UG), where we want to prove $(x)(Fx \supset Gx)$. Therefore, we assume a particular *a* such that *Fa*, deduce *Ga*, and obtain $Fa \supset Ga$ independently of the hypothesis; but since *a* was arbitrary, $(x)(Fx \supset Gx)$ follows. Hintikka rather focuses on the existential instantiation (EI): $(\exists x)Fx/Fa//p$. But both UG and EI, says Parsons, turn on "the use of a free variable which indicates *any* one of a given class of objects, so that an argument concerning it is valid for *all* objects of the class" (55). Thus, in modern logic[16], pure intuitions behave like instantiations. In fact, argues Hintikka, by instantiation methods "we introduce a representative of a particular entity a priori, without there being any such entity present or otherwise given to us" (346). This logical characterization of Kantian intuitions, he concludes, "has been misunderstood almost universally" (345).[17]

5. Quantifying over Intuitions

HP shows consistency with the quantification involved by the Kantian construction of concepts. As he identifies the possibility of conceptual constructions with the synthesis of the manifold of pure intuitions, Kant thinks of these intuitions in terms of quanta, namely quantified parts (or properties). Recall that in modern logic, binding a variable that ranges over a domain of discourse is called 'quantification.' In this sense, free variables are intended to be quantified. Since intuitions imply quanta, their logical connotation is hardly deniable.

Let's go back to the notion of object. According to Kant, an object is a collection of properties (representations). Intuitions and concepts respectively provide properties and synthetic unity (collection). Any object is, thus, determined by the conceptual synthesis of the manifold of intuitions; and the same manifold can be seen either as empirical or as pure, depending on (the presence or absence of) sensations. The material element (sensation) of such manifold derives only from experience, namely a posteriori, while the formal element of it (synthesis and pure intuitions) is already available a priori. At this point, the conceptual construction comes into play: if "an *a priori* concept [...] already includes in itself a pure intuition [...] it can be constructed" (A719/B747), otherwise not. But if it is the case, what kind of object does derive from the synthesis of such a pure manifold? What object is made by pure intuitions alone? Kant's answer is straightforward:

[16] See Thomas Lockhart, 2006.
[17] See Robert Hanna, 2001.

> As regards the formal element, we can determine our concepts in *a priori* intuition, inasmuch as we create for ourselves, in space and time, through a homogeneous synthesis, the objects themselves – these objects being viewed simply as *quanta*. (A723/B751)

From the synthesis of the pure manifold derive 'quanta', that is, objects (wholes) whose parts allow for quantification, and each of these parts corresponds to a pure intuition, which is, therefore, thought of as quantifiable from the outset. It is, then, quite consistent for Kant to hold that mathematical knowledge is "limited exclusively to quantities"; in fact, "it is the concept of quantities only that allows of being constructed, that is, exhibited *a priori* in intuition" (A714/B742). In this sense, the notion of quanta represents a building block of intuition-based constructions.

Quanta are further specified, for "mathematics does not only construct magnitudes (*quanta*) as in geometry; it also constructs magnitude as such (*quantitas*), as in algebra" (A717/B745); they instantiate ostensive and symbolic constructions in intuition, respectively. In both cases, ***a)*** quanta represent homogeneous parts proportionally combined, ***b)*** which depend on the nature of space and time, and ***c)*** which, if restricted to symbolic constructions, also account for natural numbers.

> A concept of space and time, as quanta, can be exhibited *a priori* in intuition, that is, constructed, either in respect of the quality (figure) of the quanta, or through number in their quantity only (the mere synthesis of the homogeneous manifold). (A720/B748)

a) Kant characterizes quanta by means of part-whole relationships[18] and proportions. His leading idea is that combining homogeneous parts leads to magnitudes. However, the combinations of parts require proportions (ratios)[19], since only parts that stay in proportion can be either bigger or smaller or equal. And proportions require part-whole relationships because, in order to be in proportion (bigger, smaller, equal), two parts must be seen as inside of one another, that is, as part and whole. Thus in *Metaphysics* L_2: "something is larger than the other if the latter is only equal to a part of the former" (28:561), that corresponds to: "A > than B if a part of A=B; in contrast A < B, if A is equal to a part of B" (28:506). In this sense, the homogeneity[20] of the parts is assumed for their relations

[18] See Charles Parsons, 1992[1].

[19] See Daniel Sutherland, 2004, and 2006.

[20] In *Metaphysics Vigilantius* (K_3), Kant differentiates between *quantum* and *compositum*. Both are magnitudes containing a plurality, but the latter stands for "an aggregate of heterogeneous parts", while the former accepts only homogeneous

from the outset. Therefore, "each quantum is as a multitude; each quantum must thus also consist of homogeneous parts" and, as such, "each quantum can be increased or decreased" (28:561). According to mereological combinations, "something is larger than the other if the latter is only equal to a part of the former"; thus "for something to alter into a larger is to increase, and for something to alter into a smaller is to decrease" (28:561).

b) As quantified parts, intuitions are proportionally combined in a whole. I trace these quanta back to the nature of space and time as it is presented in the *Aesthetic*. Here, Kant argues for the ideality of space and time: if abstracted from everything real (empirical and a posteriori), they are still available as ideal (pure and a priori) places. Even if they are empty, space and time are still something that can be described in mereological terms, namely wholes whose parts are introduced by arbitrary limitations and, accordingly, characterized by magnitudes. After all, "the magnitude of an intuition" is generated by "the synthesis of one and the same thing in time and space" (A724/B752), and "space is represented as an infinite *given* magnitude" (A25/B40). In this sense, I believe that the ideality of space and time already carries a logical connotation on which quanta ultimately depend.

c) If quanta are thought in succession, they represent numbers. Kant assumes that "numerus *est quantitas phenomenon*" (A146/B186), and he consistently identifies the notion of number[21] with the schema of magnitude. This schema derives from variations on quanta that imply time-series, which is required for the addition of discrete quanta. The addition is "due to my generating time itself" (A143/B182) "in the successive apprehension of an object" (A145/B184). In this case, schema and time-succession are thus intertwined: "the pure *schema* of magnitude (*quantitatis*), as a concept of the understanding, is *number*, a representation which comprises the successive addition of homogeneous units" (A142/B182). Schemata are "universal rules of synthesis" (A146/B185). They do not limit the conceptual synthesis to any individual instance, but rather consider "the subsumption of intuitions under pure concepts" (A138/B177) in general. For example, a specific concept (image) of triangle "would never attain that universality of the concept which renders it valid of all triangles,

parts. Thus, "[a] composite differs from quantum, and the many would in that case be able to be a variety, every quantum contains a multitude but not every multitude is a quantum; rather [it is one] only when the parts are homogeneous" (29:991).

[21] My analysis of this notion is limited to quanta. It is, however, worth mentioning that the idea of succession has been abandoned in modern mathematics, where claims are made about timeless structures. Frege already uses timeless relations to claim, for instance, that one set is the union of two others.

whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere" (A141/B180). In this sense, a schema represents a higher order of universality in contrast to (and in connection with) a specific universal concept.[22]

Thus, "all the various operations through which the magnitudes are produced and modified" are "in accordance with certain universal rules" (A717/B745); and these rules pertain to schemata. In the case of numbers, the schema of magnitude is the main rule for deriving magnitudes; it employs both conceptual syntheses and time-series. These syntheses correspond to the categories of quantity (*unity*, *plurality*, *totality*) and are so employed that they establish a unit, repeat the unit, and sum the repeated units; they, claims Longuenesse, allow for "the repetition of unit as many times as one wishes" (1998: 267). Similar conclusions are reached by Guyer (1987: 190-6). In this sense, the *Prolegomena* explicitly refer quantity to measure;[23] here, in fact, the categories of *unity, plurality*, and *totality* correspond to *measure*, *magnitude*, and *the whole*, respectively. Consistently, this schema of quantity explains numbers in terms of units synthesized in successive addition; and, since synthesized units are discrete quanta, numbers are derived from the series of quanta, namely quantified intuitions.

A unit is established by introducing a limitation in continuous quanta, which are represented by space and time. Kant identifies two kinds of quanta: "a quantum continuous in itself is one in which the number of parts is indeterminate; a quantum discrete in itself is one in which the number of parts is arbitrarily determined by us" (28:561). Therefore, units correspond to discrete quanta and, consistently, behave like "assignable parts" within the field of continuous quanta. Space and time are thus subject to arbitrary partitions, namely numerations: "discrete quantum is therefore called number" because "through number we represent each quantum as discrete" (28:561). In this sense, a number is clearly conceived as extensive magnitude, namely a magnitude that allows for mereological

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[22] Recall that a set formalizes the idea of grouping objects together and views them as single entity. It may be any well-defined collection of objects (elements or members), which is specified either in a tabular form by listing its elements: $A=\{a, e, i, o, u\}$, or in a set-builder form by stating its properties or rules: $B=\{x : x \text{ is an even integer}, x>0\}$. Now, if we stipulate that categories behave like logical operators, then we need symbols that stand for variables or constants together with a property method of specifying the set. Intuitions can provide the former, and schemata can provide the latter. In this sense, a schema can be seen as a rule for arranging symbols via categories.

[23] See Gary Hatfield, 2006.

partitions;[24] and, according to Kant, "all intuitions are extensive magnitudes" (B202).

In conclusion, the Kantian notion of quanta clearly derives numbers from quantifications over intuitions. Therefore, the only connotation that properly characterizes these intuitions is logical. Consistently, HP is confirmed.

6. Intuitions and Mathematical Induction

Further evidence for HP comes from mathematical induction, a kind of reasoning that is usually neglected by Kant scholars.

Kant believes (but many don't) that mathematics exemplifies "the method of attaining apodeictic certainty" (A713/B741), if there is any. In fact, the first *Critique* inquires into the nature of synthetic a priori claims because they represent a pattern for mathematical claims. They look like the right way to "combine some of the features of empirical propositions with some of the features of analytic propositions" (Strawson, 1966: 277). In synthetic a priori claims, the certainty that comes a priori from analysis meets the knowledge that derives a posteriori from synthesis. However, I believe that in these claims, 'a priori' and 'a posteriori' do not pertain to the synthesis but rather characterize its justification.[25] If I am right, the part played by Kantian intuitions in mathematical concepts resembles the role of variables in mathematical induction.[26]

Mathematical induction (MI) is a formal proof whose aim is to show that LHS (left hand side) is equal to RHS (right hand side). The proving process runs in a few steps: ***1)*** show the statement is true for the initial

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[24] Thus in *Metaphysics* L_2: "All magnitudes (quantities) can be considered in two ways: either extensively or intensively. There are objects in which we distinguish no multitude of homogeneous parts; this is *intensive* magnitude. This magnitude is the degree. The objects in which we distinguish a multitude of homogeneous parts have *extensive* magnitude. The intensive magnitude is the magnitude of the degree, and the extensive magnitude is the magnitude of the aggregate." (28:562) The same argument appears in the first *Critique*: "All combination (*conjunctio*) is either composition (*compositio*) or connection (*nexus*). The former is the synthesis of the manifold where its constituents do not necessarily belong to one another. […] Such also is the synthesis of the *homogeneous* in everything which can be *mathematically* treated. This synthesis can itself be divided into that of *aggregation* and that of *coalition*, the former applying to *extensive* and the latter to *intensive* quantities. The second mode of combination (*nexus*) is the synthesis of the manifold so far as its constituents *necessarily belong to one another*." (B201)

[25] See Philip Kitcher, 2006.

[26] See David S. Gunderson, 2011.

value of n (base case); ***2)*** assume the statement is true for $n=k$ (assumption); ***3)*** prove the statement is true for $n=k+1$ (induction); and ***4)*** assert that statement must be true for all values of n (conclusion).

Let's apply the proof to solve an exemplary question. Prove $2+4+6+...+2n=n(n+1)$ by MI. In this case, $2n$ is a general formula for each of the values (2, 4, 6...) up to $2n$: establishing $n=1$ gives the value of 2 (first value), $n=2$ gives 4 (second value), and so on to $2n$ that gives us the last term; therefore, $n=2$ gives us the value of each term.

1) Show it is true for $n=1$ (initial value of n). For the general formula ($2n$) to equal to the first value, n has to be 1. In fact, LHS=2•1=2 is equal to RHS=1•(1+1)=2. Therefore, it is true for $n=1$.

2) Assume that it is true for $n=k$. By substituting n with k, $2+4+6+...+2k=k(k+1)$.

3) Show it is true for $n=k+1$, that is $2+4+6+...+2k+2(k+1)=(k+1)(k+1+1)$. Since the series must be continuous, numbers increase or decrease by 1 (in terms of n): given ...$2k-2$, $2k-1$, $2k$, $2k+1$, $2k+2$..., $2k$ derives from $2(k+1-1)$; in other words, the initial value of n must *be included*. As assumed (**2**), $2+4+6+...+2k=k(k+1)$; therefore, LHS $= k(k+1)+2(k+1) = (k+1)(k+2)$ is equal to RHS $= (k+1)(k+2)$.

4) The LHS and RHS are equal, therefore $2+4+6+... +2k+2(k+1)=(k+1)(k+1+1)$ is proved; then it is true that $n=k+1$. In ***1***, we proved $n=1$; in ***4***, we proved $n=k+1$, i.e., that n is true for any number +1. In conclusion, the statement is true for $n\geq 1$.

MI is a formal proof that establishes a given statement for all natural numbers. Two things must be noticed. First, though it is limited to mathematics, MI brings new knowledge; and, second, MI relies on variables throughout the inductive process. The assumption (step ***2***) for induction (step ***3***) precisely consists in the introduction of a variable of higher level (k) entailing a higher-order logic (Parsons, 1992[3]: 140). There is no other way to prove with certainty a chain of inductive-based relations. In conclusion, variables (n, k) are required for a synthesis a priori to call for certain inductive proofs (MI); and Kant explicitly identifies the role of these variables with pure intuitions. Therefore, HP is directly confirmed by MI.

7. Conclusions

Let's draw a few conclusions. ***a)*** Intuitions are immediate representations of objects by Kantian definition. In this sense, they are references that provide evidence for our semantics. Our judgments rely on this evidence insofar as they aim to justify the relations of *S*-term and *P*-term that they claim. If *S*-term and *P*-term are intended as class relations, intuitions ultimately show their inclusion or exclusion. Sentences like 'the apple is green' or 'every part belongs to a whole' exemplify synthetic and analytic judgments, respectively symbolized by $(\exists x)(Sx \bullet Px)$ and $(x)(Sx \supset Px)$; they thus represent particular and universal claims, which can be reduced to relations of class intersection and class inclusion, namely $S \cap P$ and $S \supset P$. This may explain the cognitive function of intuitions according to any classic theory of judgments. Kant starts here.

b) But, Kantian intuitions are still something, even without reference to an empirical object. Any specific collection of properties (empirical intuitions) mirrors the conceptual unity of pure intuitions, the nature of which is specified as ideal and mereological. Therefore, each of these intuitions behaves like a placeholder because it can stand for (be filled out by) any sensation whatsoever – Kant reasons via EG: if $\boldsymbol{F}a$ or $\boldsymbol{F}y$ then $(\exists x)\boldsymbol{F}x$. In this sense, intuitions are not only references for our semantics, they also correspond, as HP argues, to free variables for our formal semantics: Kant, indeed, thinks of them as individualities that stand for universalities, and grounds both the notions of synthetic a priori and quanta on them. But these intuitions can be logically categorized or quantified only if their nature is intended to be logical, and textual evidence from Kant clearly supports this interpretation. ***c)*** Furthermore, empirical and pure intuitions share the same synthesis but cannot appeal to the same evidence: the former call for experience, the latter do not. I believe, therefore, that 'a priori' and 'a posteriori' may pertain to the justification rather than the synthesis of Kantian judgments. In fact, if we look at the case of MI, we see a proving process that, independent of experience, guarantees those results that Kant allegedly attributes to synthetic a priori judgments, namely certainty and knowledge. MI and Kant, respectively, rely on variables and intuitions, which, thus, share the same function and nature.

HP represents a radical breakthrough that turns Kant's theoretical philosophy into mathematical intuitionism. After centuries of attempts, HP finally shows us how to read Kantian intuitions properly, and it also suggests a way to rethink Brouwer's (1913) understanding of the relation between formalism and intuitionism, which, I believe, is worth investigating.

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